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THE NON-STATIONARY INVARIANT SOLUTION OF THE EQUATIONS OF GAS DYNAMICS DESCRIBING THE SPREADING OF A GAS INTO A VACUUM*

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An invariant solution of the equations of gas dynamics, constructed on a one-dimensional subgroup (according to the classification in /1/) which is only allowed in the case of a polytropic gas with a special adiabatic index, is considered. The gas spreads out into a vacuum after a finite time. New solutions are constructed which describe one-dimensional flows from a source into a vacuum and the focussing of the gas within a sphere or a cylinder with shock waves. The spreading of a concentration of the gas with an arbitrary boundary when there is a contact discontinuity is also considered.

One-dimensional flows have been treated in detail in /2, 3/, mainly in the case of extended subgroups.

1. Equations for the invariant solution. The equations of gas dynamics in the case of a polytropic gas with an equation of state

$$p = A(S)\rho^\gamma,\ A(S) = \exp[(S - S_0)/{}^1/_2 nR]$$

and a special adiabatic index $\gamma = (n+2)/n$ in a space $x_1, x_2, \ldots, x_n$ have the form

$$\frac{d\mathbf{u}}{dt} + \frac{1}{\rho}\nabla p = 0,\quad \frac{d\rho}{dt} + \rho\,\mathrm{div}\,\mathbf{u} = 0,\quad \frac{dp}{dt} + \gamma p\,\mathrm{div}\,\mathbf{u} = 0 \tag{1.1}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ is the velocity, ρ is the density, p is the pressure, S is the entropy and R is the gas constant. Instead of the last equation, one may take $dS/dt = 0$.

System (1.1) permits a $({}^1/_2 n(n+3) + 5)$-parameter group of point transformations (/4/, p.146). In doing this, there is a special operator which is only permited in the case of an index $\gamma = (n+2)/n$. When $n = 2$, all the dissimilar subgroups are tabulated in /1/. Next, invariant solutions are considered which are constructed on a family of one-dimensional subgroups of this classification containing a special operator but for any natural n. A subgroup is specified by the operator

$$H_\alpha = (1 + \alpha t + t^2)\frac{\partial}{\partial t} + (t + \alpha)x_i\frac{\partial}{\partial x_i} + (x_i - tu_i)\frac{\partial}{\partial u_i} - n t\rho\frac{\partial}{\partial \rho} - (n+2)tp\frac{\partial}{\partial p}$$

where the subgroup parameter $0 \leqslant \alpha < 2$.

An invariant solution, constructed on a subgroup H_α, is written in the following form (c is the velocity of sound):

$$\begin{gathered}\mathbf{u} = e^{\theta(t)}\frac{\bar{\mathbf{u}} + t\bar{\mathbf{r}}}{(1 + \alpha t + t^2)^{1/2}},\quad \rho = e^{n\theta(t)}\frac{\bar\rho}{(1 + \alpha t + t^2)^{n/2}} \\ p = e^{(n+2)\theta(t)}\frac{\bar p}{(1 + \alpha t + t^2)^{n/2+1}},\quad \bar S = S,\quad c^2 = e^{2\theta(t)}\frac{\bar c^2}{1 + \alpha t + t^2} \\ \bar p = A(\bar S)\bar\rho^{(n+2)/n},\quad c^2 = \frac{n+2}{n}\frac{p}{\rho},\quad \bar c^2 = \frac{n+2}{n}\frac{\bar p}{\bar\rho} \\ \mathbf{r} = \bar{\mathbf{r}}e^{\theta(t)}(1 + \alpha t + t^2)^{1/2},\quad \theta(t) = \frac{\alpha}{\sqrt{4 - \alpha^2}}\,\mathrm{arctg}\,\frac{2t + \alpha}{\sqrt{4 - \alpha^2}}\end{gathered} \tag{1.2}$$

The functions $\bar\rho$, $\bar p$, $\bar c$, $\bar S$ and $\bar{\mathbf{u}} = (\bar u_1, \ldots, \bar u_n)$ depend on $\bar{\mathbf{r}} = (\bar x_1, \ldots, \bar x_n)$.

Let $\bar{\mathbf{u}}, \bar\rho, \bar p$ and $\bar c$ be finite at any instant of time. Then, as $t \to \infty$, the gas parameters $\mathbf{u}, \rho, p$ and c approach values corresponding to the vacuum state and to retardation at all points of the space. When $t = 0$, we get

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$$\mathbf{r}_0 = \mathbf{r}e^{-\theta_0}, \quad \mathbf{u}(0, \mathbf{r}) = e^{\theta_0}\bar{\mathbf{u}}(\mathbf{r}e^{-\theta_0})$$
$$\rho(0, \mathbf{r}) = e^{n\theta_0}\bar{\rho}(\mathbf{r}e^{-\theta_0}), \quad p(0, \mathbf{r}) = e^{(n+2)\theta_0}\bar{p}(\mathbf{r}e^{-\theta_0})$$
$$\theta_0 = \frac{\alpha}{\sqrt{4-\alpha^2}} \operatorname{arctg} \frac{\alpha}{\sqrt{4-\alpha^2}}$$

This means that the values of $\bar{\mathbf{u}}, \bar{\rho}$ and $\bar{p}$ are practically the initial parameters of the special gas.

Substitution of (1.2) into (1.1) yields the system of the H_α-invariant solution which is similar to the system for the steady state motion of the gas in a centrosymmetric gravitational field which is proportional to the radius

$$\begin{aligned} &(\bar{\mathbf{u}} - \alpha\bar{\mathbf{r}})\cdot\nabla\bar{\mathbf{u}} + \bar{\rho}^{-1}\nabla\bar{p} + \bar{\mathbf{r}} = 0 \\ &(\bar{\mathbf{u}} - \alpha\bar{\mathbf{r}})\cdot\nabla\bar{\rho} + \bar{\rho}\operatorname{div}\bar{\mathbf{u}} = 0 \\ &(\bar{\mathbf{u}} - \alpha\bar{\mathbf{r}})\cdot\nabla\bar{p} + ((n+2)/n)\,\bar{p}\operatorname{div}\bar{\mathbf{u}} = 0 \;\text{ or }\; (\bar{\mathbf{u}} - \alpha\bar{\mathbf{r}})\cdot\nabla S = 0 \end{aligned} \tag{1.3}$$

The line L with the tangential direction $\bar{\mathbf{u}} - \alpha\bar{\mathbf{r}}$ at each point is called a stream line. The entropy integral $\bar{S} = S_0(L)$ is valid along it.

When $\alpha = 0$ or in the case of flows when $\bar{\rho} = -\bar{r}^{-2}\bar{\mathbf{r}}\cdot\nabla\bar{p}$, $\bar{r} = |\bar{\mathbf{r}}|$, the Bernoulli integral

$$\bar{q}^2 + n\bar{c}^2 + \bar{r}^2 = D^2(L), \quad \bar{q} = |\bar{\mathbf{u}}| \tag{1.4}$$

is valid.

In this case the flow takes place in a sphere $\bar{r} \leqslant D$, on the boundary of which a vacuum and retardation are attained. In the case of a fixed $\bar{r}$, the critical velocity is defined by $\bar{c}_*^2 = (D^2 - \bar{r}^2)/(n+1)$.

The Bernoulli integral is valid in the case when $\bar{q} = \alpha\bar{r}\cos\beta$, where β is the angle between the vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{r}}$ and, also, in the case of potential flows.

It has been verified that, if the flow in isentropic, it is also iso-energetic, that is, the constant in the Bernoulli integral is independent of the stream line.

The characteristics of system (1.3) are sought in the form $h(\bar{\mathbf{r}}) = 0$. If $\mathbf{n}$ is a unit normal, then one characteristic coincides with the streamline $(\bar{\mathbf{u}} - \alpha\bar{\mathbf{r}})\cdot\mathbf{n} = 0$ while the other two have the form $(\bar{\mathbf{u}} - \alpha\bar{\mathbf{r}})\cdot\mathbf{n} = \pm\bar{c}$ and, when $\alpha = 0$, they are only possible when $\bar{q} > \bar{c}$.

Let the surface of a strong discontinuity $F(\mathbf{r}, t) = 0$ be H_α-invariant. Then, it has the form $F(\bar{\mathbf{r}}) = 0$. The velocity of the surface in the direction of the normal $\mathbf{n}$ is:

$$D_n = -\frac{F_t}{\mathbf{n}\cdot\nabla F} = \bar{D}_n \frac{(t+\alpha)e^\theta}{(1+\alpha t+t^2)^{1/2}}, \quad \bar{D}_n = \bar{\mathbf{r}}\cdot\mathbf{n} \tag{1.5}$$

A contact discontinuity is characterized by the equalities $u_n = D_n$, $[p] = 0$, $[u_n] = 0$ (the square brackets denote a jump in the quantities inside the brackets). These equalities are written in the H_α-invariants

$$\bar{u}_n = \alpha\bar{D}_n, \; [\bar{p}] = 0, \; [\bar{u}_n] = 0 \tag{1.6}$$

The conditions on the shock wave (SW) are specified by the equalities

$$[\mathbf{u}_\sigma] = 0, \; [\rho v] = 0, \quad [p + \rho v^2] = 0, \; [(n+2)p/\rho + v^2] = 0,$$
$$v = u_n - D_n$$

where $\mathbf{u}_\sigma$ is the projection of the vector $\mathbf{u}$ on to the tangent plane to the surface of the explosion.

The relationships

$$\begin{aligned} &\bar{v}_2 = \frac{1}{n+1}\left(\bar{v}_1 + n\frac{\bar{c}_1^2}{\bar{v}_1}\right), \\ &\bar{c}_2^2 = \frac{1}{(n+1)^2}\left(\bar{v}_1 + n\frac{\bar{c}_1^2}{\bar{v}_1}\right)\left((n+2)\bar{v}_1 - \frac{\bar{c}_1^2}{\bar{v}_1}\right) \quad \bar{\mathbf{v}} = \bar{\mathbf{u}}_n - D_n \end{aligned} \tag{1.7}$$

where the indices 1 and 2 denote the states of the gas along the different sides of the shock wave, follow from these conditions which have been written in terms of H_α-invariants.

2. The group property of the invariant motion. System (1.3) permits a $[{}^1/_2 n(n-1)]$-dimensional group of rotations, Z_i and two extension operators

$$X = x_i\frac{\partial}{\partial x_i} + \bar{u}_i\frac{\partial}{\partial \bar{u}_i} + 2\bar{p}\frac{\partial}{\partial \bar{p}}, \quad Y = \bar{\rho}\frac{\partial}{\partial\bar{\rho}} + \bar{p}\frac{\partial}{\partial\bar{p}}$$

The operator Y forms the centre of the whole algebra and it may therefore be added with an arbitrary coefficient to any operator of any subalgebra. Taking account of the remark which has just been made, we shall write the system of dissimilar subgroups when $n > 1$ in the form /1, 5/

$$\langle Z+\beta X\rangle,\ \langle X\rangle;\ \ \langle X,\ Z\rangle;\ \ \langle Z_1,\ Z_2,\ Z_3\rangle;\ \ \langle Z_1,\ Z_2,\ Z_3,\ X\rangle$$

The invariant solutions can be considered on each of these subgroups. For example, calculation of the invariants of the subgroup $Z+\beta X+\gamma Y$ when $n=3$ leads to the following form for the invariant solution:

$$u_1 = e^{\beta\varphi}(U\cos\varphi - V\sin\varphi),\ u_2 = e^{\beta\varphi}(U\sin\varphi + V\cos\varphi)$$
$$u_3 = e^{\beta\varphi}W,\quad \rho = e^{\gamma\varphi}R,\quad p = e^{(\gamma+2\beta)\varphi}P \tag{2.1}$$

where U, V, W, R and P depend on $\bar{s} = se^{-\beta\varphi}$, $\bar{z} = x_3e^{-\beta\varphi}$; $x_1 = s\cos\varphi$, $x_2 = s\sin\varphi$. Substitution into (1.3) leads to the system for determining U, V, W, R and P.

System (1.3) has a symmetric solution of the form

$$\bar{\mathbf{u}} = \bar{r}^{-1}\bar{q}(\bar{r})\,\bar{\mathbf{r}},\quad \bar{\rho}(\bar{r}),\quad \bar{p}(\bar{r}),\quad \bar{c}^2 = ((n+2)/n)\,\bar{p}/\bar{\rho}$$

with respect to the origin of the coordinate system which is a particular invariant solution constructed on a subgroup of rotations and enters into the class of one-dimensional motions of a gas with spherical $(n=3)$, cylindrical $(n=2)$ and planar $(n=1)$ waves. The functions $\bar{q}$ and $\bar{c}$ satisfy the system

$$\bar{q}'(\bar{q}-\alpha\bar{r}) + n\bar{c}\bar{c}' + \bar{r} = 0$$
$$n\bar{c}'(\bar{q}-\alpha\bar{r}) + \bar{c}(\bar{q}' + (n-1)\bar{q}/\bar{r}) = 0 \tag{2.2}$$

When $\alpha = 0$, there are two integrals which completely describe the motion

$$\bar{q}^2 + n\bar{c}^2 + \bar{r}^2 = D^2,\quad \bar{q}\bar{c}^n\bar{r}^{n-1} = E \tag{2.3}$$

When $\alpha \neq 0$, system (2.2) can be written in the parametric form

$$d\bar{q}/dl = (n-1)\bar{q}\bar{c}^2 - \bar{r}^2(\bar{q}-\alpha\bar{r})$$
$$d\bar{c}/dl = n^{-1}\bar{c}\,[\bar{r}^2 - (n-1)\bar{q}(\bar{q}-\alpha\bar{r})]$$
$$d\bar{r}/dl = \bar{r}\,[(\bar{q}-\alpha\bar{r})^2 - \bar{c}^2] \tag{2.4}$$

The substitution $\bar{q} = \bar{r}\,(Q(s)+\alpha)$, $\bar{c} = \bar{r}\,C(s)$, $s = \ln\bar{r}$ leads to the selfcontained system of the two equations

$$\frac{dQ}{ds} = \frac{-Q + (Q+\alpha)(nC^2 - Q^2)}{Q^2 - C^2},$$
$$n\frac{dC}{ds} = C\,\frac{1 + n(C^2 - Q^2) - (n-1)Q(Q+\alpha)}{Q^2 - C^2} \tag{2.5}$$

By dividing the second of these two equations by the first, we get

$$\frac{dC}{dQ} = \frac{1}{n}C\,\frac{1 + n(C^2 - Q^2) - (n-1)Q(Q+\alpha)}{-Q + (Q+\alpha)(nC^2 - Q^2)} \tag{2.6}$$

The solutions of system (2.2) are studied in detail later and it is explained to which problems they correspond in gas dynamics.

3. One-dimensional motions when $\alpha = 0$. When $n = 1$, it follows from (2.3) that $\bar{q} = E\bar{c}^{-1}$, $\lambda = \bar{c}^2$

$$F(\lambda) = \lambda + E^2\lambda^{-1} = D^2 - \bar{r}^2 = G(\bar{r}) > 0 \tag{3.1}$$

Plots of the functions $F(\lambda)$ and $G(\bar{r})$ qualitatively describe the motion of the gas. Let the maximum of $G(\bar{r})$ be greater than the minimum of $F(\lambda)$, that is, $D^2 > 2\,|E|$. Then, the minimum of $G(\bar{r})$ defines the domain of the flow $0 \leqslant \bar{r} \leqslant r_1 = (D^2 - 2\,|E|)^{1/2}$. At the minimum point, $\lambda = |E|$ satisfies the equality $|q| = \bar{c} = |E|^{1/2}$ while the velocity of sound is attained on the boundary of the flow domain $\bar{r} = r_1$. The function $F(\lambda)$ is defined on the set $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, where $\lambda_{2,1} = {}^1/_2[D^2 \pm (D^4 - 4E^2)^{1/2}]$. The subsonic velocities $|\bar{q}| < |E|^{1/2} < \bar{c}$ correspond to the values of λ in the interval $|E| < \lambda < \lambda_2$ while the supersonic velocities $|\bar{q}| > |E|^{1/2} > \bar{c}$ correspond to the values of λ in the interval $\lambda_1 < \lambda < |E|$. Hence, two flows correspond to the two branches of the function $\lambda(F)$: one is subsonic and the other is supersonic. Moreover, there is a source $(\bar{q} > 0,\ E > 0)$ or a sink $(\bar{q} < 0, E < 0)$, at the point $\bar{r} = 0$ and a further acoustic sink or source at the point $\bar{r} = r_1$.

The solution when $E = 0$ is referred to as retardation.

$$\bar{q}_0 = 0,\ \bar{c}_0^{\,2} = D_0^{\,2} - \bar{r}^2 \tag{3.2}$$

It is defined in the domain $0 \leqslant \bar{r} \leqslant D_0$. A vacuum state is attained when $\bar{r} = D_0$.

Theorem 1. The solution of (3.1) with the condition

$$D^2 > 4\,|E| > 0 \tag{3.3}$$

is coupled with the solution of (3.2) through a shock wave with a specified direction for its velocity, if the inequalities $\sqrt{2}D_0 > D > D_0/\sqrt{2}$ are satisfied. Generally, speaking, two possible shock transitions exist.

Proof. The velocity of the shock wave is equal to $\pm\bar{r}$. On a shock transition, conditions (1.7) take the form

$$\bar{q} = \pm\left(\bar{r} - \frac{D_0^2}{2\bar{r}}\right), \quad \bar{c}^2 = D_0^2\left(1 - \frac{D_0^2}{4\bar{r}^2}\right)$$

The place where the shock wave is located, $\bar{r} = r_* = D/\sqrt{2}$ is found from (3.1). This means that

$$\bar{q} = \frac{D^2 - D_0^2}{\sqrt{2}\,D}, \quad \bar{c}^2 = D_0^2\left(1 - \frac{D_0^2}{2D^2}\right) > 0$$

The equality $\bar{q}\bar{c} = E$ is equivalent to the equation

$$f(\varkappa) = \varkappa(2 - \varkappa)(1 - \varkappa)^2 = \delta, \ \varkappa = D_0^2 D^{-2}, \ \delta = 4E^2 D^{-4} \tag{3.4}$$

the roots of which have to be sought in the interval $0 < \varkappa < 2$. The function $f(\varkappa)$ has a minimum equal to zero at the point $\varkappa = 1$ and two maxima equal to $^1/_4$ at the points $\varkappa_\pm = 1 \pm 1\sqrt{2}$. The inequality (3.3) follows from this and ensures that the shock wave is located in the flow domain $r_* < r_1$.

When $\bar{q} > 0$, if the velocity of the shock wave is equal to $\bar{r}$ (or $-\bar{r}$), then, according to Zemplén's theorem, retardation occurs in front of (or behind) the shock wave front and the point source is coupled with a vacuum.

When $\bar{q} < 0$, retardation occurs behind (in front of) the shock wave front and the gas is focussed within the domain $\bar{r} < r_1$.

There are two roots of Eq.(3.4) subject to condition (3.3) in each of the intervals $1 < \varkappa < 2$ $0 < \varkappa < 1$, which correspond to the two possible shock transitions for a specified direction of the velocity of the shock wave. When $\varkappa < 1$, a unique solution is chosen with the aid of the inequality $r_* < D_0$. The root of Eq.(3.4) then lies in the interval $\varkappa_- < {}^1/_2 < \varkappa < 1$.

When $n = 2$, it follows from (2.3) that

$$\bar{q} = E\bar{r}^{-1}\bar{c}^{-2}, \quad \lambda = \bar{c}^2\bar{r}^{2/3}$$
$$F(\lambda) = 2\lambda + E^2\lambda^{-2} = \bar{r}^{2/3}(D^2 - \bar{r}^2) = G(\bar{r}) \tag{3.5}$$

Let the minimum of the function $F(\lambda)$ be less than the maximum of $G(\bar{r})$, that is,

$$|E| < (^1/_2 D)^4 \tag{3.6}$$

The flow domain is then defined using the minimum of the function $F(\lambda)$: $r_1 < \bar{r} < r_2$, where r_i, are the roots of the equation $3^3E^2 = \bar{r}^2(D^2 - \bar{r}^2)^3$. The velocity of sound $|\bar{q}_i| = \bar{c}_i = |E|^{1/3}\, r_i^{-1/3}$ is attained on the boundaries of this domain. Consequently, $\bar{r} = r_i$ defines the boundaries of the non-point acoustic sources or sinks. The maximum of the function $G(\bar{r})$ specifies the domain of definition of the function $F(\lambda): \lambda_1 < \lambda < \lambda_2$, where $\lambda_i > 0$ are the roots of the equation $3(^1/_2 D)^{4/3} = 2\lambda + E^2\lambda^{-2}$. In the domain $\lambda_1 < \lambda < E^{2/3}$, the flow is subsonic while it is supersonic in the domain $E^{2/3} < \lambda < \lambda_2$.

The solution when $E = 0$ is referred to as a retardation.

$$\bar{q}_0 = 0, \quad \bar{c}_0^2 = {}^1/_2(D_0^2 - \bar{r}^2) \tag{3.7}$$

It is defined in the interval $0 \leqslant \bar{r} \leqslant D_0$ and reaches the vacuum when $\bar{r} = D_0$.

The following theorem is proved in precisely the same manner as when $n = 1$.

Theorem 2. The solution of (3.5) subject to condition (3.6) is coupled with the solution of (3.7) through the shock wave with a specified direction for its velocity, if the inequalities $3D/\sqrt{5} > D_0 > \sqrt{3/7}\,D$ are satisfied. At the same time, if $\sqrt{3/7}\,D < D_0 < D$, then there exists a unique shock transition point while, if $D < D_0 < 3D/\sqrt{5}$, two shock transitions are possible.

The solutions contructed in Theorem 2 describe the flow of a gas from a non-point source into a vacuum or the focussing of the gas on the axis of a cylinder.

When $n = 3$, it follows from (2.3) that

$$\bar{q} = E\bar{r}^{-2}\bar{c}^{-3}, \quad \lambda = \bar{c}^2\bar{r}$$
$$F(\lambda) = 3\lambda + E^2\lambda^{-3} = \bar{r}(D^2 - \bar{r}^2) = G(\bar{r}) > 0 \tag{3.8}$$

The function $y = F(\lambda)$ has the asymptotes $\lambda = 0,\ y = 3\lambda$ and a minimum equal to $4|E|^{1/2}$ at the point $\lambda = |E|^{1/2}$. When $0 < \bar{r} < D$, the function $y = G(\bar{r})$ has a single maximum, equal to $2D^3/3^{3/2}$ at the point $\bar{r} = D/\sqrt{3}$. Let the minimum of $F(\lambda)$ be less than the maximum of $G(\bar{r})$, that is,

$$|e| = |E|^{1/2}/D^3 < (2\cdot 3^{3/2})^{-1} \tag{3.9}$$

The flow domain is defined using the minimum of the function $F(\lambda)$: $r_1 < \bar{r} < r_2$, where r_i are the roots of the equation $4|E|^{1/2} = \bar{r}(D^2 - \bar{r}^2)$. The velocity of sound $|\bar{q}_i| = \bar{c}_i = |E|^{1/4} r_i^{-1/2}$ is attained on the boundaries of this domain. This means that $\bar{r} = r_i$ are the boundaries of the non-point acoustic sources and sinks.

Corresponding to the two branches of the function $\lambda(F)$, two values of λ correspond to each $\bar{r}$ from the flow domain. The two branches are defined on the set $4|E|^{1/2} \leqslant F \leqslant 2 \times 3^{-3/2} D^3$ and $\lambda_1 \leqslant \lambda \leqslant |E|^{1/2}$ is the domain of values for the first branch while $|E|^{1/2} \leqslant \lambda \leqslant \lambda_2$ is the domain of values for the second branch, where $\lambda_i > 0$ are the roots of the equation $E^2\lambda^{-2} + 3\lambda = 2 \times 3^{-3/2} D^3$. If $\lambda > |E|^{1/2}$, then $\bar{q}^2 < |E|^{1/2} \bar{r}^{-1} < \bar{c}^2$ and a subsonic flow is obtained. If $\lambda < |E|^{1/2}$, then $\bar{q}^2 > |E|^{1/2} \bar{r}^{-1} > \bar{c}^2$ and a supersonic flow results. An extremal velocity is attained at the point $\bar{r} = D/\sqrt{3}$.

The solution when $E = 0$ is referred to as retardation.

$$\bar{q}_0 = 0, \ \bar{c}_0^2 = {}^1/_3 (D_0^2 - \bar{r}^2) \tag{3.10}$$

It is defined in the interval $0 \leqslant \bar{r} \leqslant D_0$ and, when $\bar{r} = D_0$, reaches the vacuum.

Theorem 3. The solution of (3.8) subject to the condition $-3^{-9/4} < e < {}^1/_2 3^{-3/4}$ is coupled with the solution of (3.10) through a shock wave with a specified direction for its velocity, if the inequalities $\sqrt{{}^2/_5} D < D_0 < 2\sqrt{{}^2/_5} D$ are satisfied. Generally speaking, two shock transitions are possible.

Proof. The velocity of the shock wave is equal to $\pm \bar{r}$. It follows from conditions (1.7) on the shock transition that

$$\bar{q} = \pm\left(\bar{r} - \frac{1}{4} D_0^2 \bar{r}^{-1}\right), \quad \bar{c}^2 = \frac{1}{48} D_0^2 (16 - D_0^2 \bar{r}^{-2}) \tag{3.11}$$

Substitution of these equalities into (3.8) yields

$$(D_0/D)^2 = 2(1 - \varkappa), \quad \varkappa = 2(\bar{r}/D)^2 \tag{3.12}$$

The second equality of (2.3) becomes the equation for determining the position $\varkappa_* = 2(r_*/D)^2$ where the shock wave is located:

$$f(\varkappa) = (2\varkappa - 1)[(1 - \varkappa)(5\varkappa - 1)]^{3/2} = k\varkappa, \ k = \pm 12^{3/2} e^2 \tag{3.13}$$

By virtue of (3.9), $|k| < k_0 = 2 \times 3^{-3/2}$ and, by virtue of (3.11) and (3.12), the function $f(\varkappa)$, defined in the interval ${}^1/_5 < \varkappa < 1$, has a negative minimum and a positive maximum at the points

$$\varkappa_\pm = \frac{1}{16}\left(9 \pm \sqrt{\frac{53}{5}}\right), \quad f(\varkappa_+) = 0.29, \quad f(\varkappa_-) = -0.1$$

and three zeros at the points ${}^1/_5$, ${}^1/_2$ and 1. The two functions, $f(\varkappa)$, which are tangential to the graph at the points $\varkappa_0 = {}^1/_3 < \varkappa_-$, $\varkappa_0' = (3 + \sqrt{19})/10$, pass through the origin of coordinates and have the angular coefficients $f'(\varkappa_0) = -8 \times 3^{-3} > k_0$, $f'(\varkappa_0') = 0.42 > k_0$. This means that a shock transition is only possible for values of k which satisfy the inequalities $-8 \times 3^{-3} < k < 2 \times 3^{-3/2}$. The inequalities involving e, which have been presented in the formulation of this theorem, follow from this. For such k, Eq.(3.13) has two roots to which the two possible shock transitions correspond.

The shock wave must be located in the domain of the flow $r_* < D_0$. It follows from this that $\varkappa_+ < \varkappa_* < {}^4/_5$ and a single shock transition is possible if $0 < k < {}^5/_4 f({}^4/_5) < k_0$ and, at the same time, ${}^1/_2 < \varkappa_* < a < {}^4/_5$, where ${}^5/_4 a f({}^4/_5) = f(a)$. The shock wave must be located in the domain of the flow (2.3): $r_1 < r_* < r_2$ or $\varkappa_1 < \varkappa_* < x_2$, where $\varkappa_i = 2(r_i/D)^2$ satisfies the equation

$$|k|\varkappa = (3^{3/2}/16)\varkappa^2(2 - \varkappa)^2 = g(\varkappa)$$

This means that the inequality $g(\varkappa) > |f(\varkappa)|$ must be satisfied, which is confirmed by the calculations.

According to Zemplén's theorem, when $\varkappa_* < {}^1/_2$ $(\varkappa_* > {}^1/_2)$, the state (2.3) is found ahead of (behind) the shock wave front. If the velocity of the shock wave is equal to $r_*, \varkappa_* > {}^1/_2$ $(-r_*, \varkappa_* < {}^1/_2)$, then $\bar{q} > 0$ $(\bar{q} < 0)$ and the gas source with the boundary $\bar{r} = r_1$ is coupled through the shock wave at $\bar{r} = r_*$ with a vacuum at $\bar{r} = r_0$. If the velocity of the shock wave is equal to $-r_*$, $\varkappa_* > {}^1/_2$ $(r_*, \varkappa_* < {}^1/_2)$, then $\bar{q} < 0$ $(\bar{q} > 0)$ and the gas is focussed within the sphere $\bar{r} = r_2$.

4. One-dimensional motions when $\alpha \neq 0$. Let us investigate the behaviour of the integral curves of Eq.(2.6) in the half-plane $C > 0$ /6/.

There are two singular points when $n = 1$: one at the origin of the coordinate system (a saddle point) and the other at an infinitely remote point. The curve which enters the saddle point has the asymptotic behaviour $Q = {}^1/_3 \alpha C^2 + O(C^4)$ when $C \to 0$. The other end of this curve enters into the infinitely remote singular point. Hence, the curve joining the singularities separates the remaining integral curves which begin and terminate at an infinitely remote point.

System (2.4) is integrated when $n = 1$. After making the substitutions $v = \bar q - \alpha\bar r + \bar c$, and $w = \bar q - \alpha\bar r - \bar c$, the variables v and w are separated:

$$v^2 + \alpha v\bar r + \bar r^2 = V^2 \exp\left[\frac{\alpha}{\Delta}\operatorname{arctg}\left(\frac{v}{\Delta\bar r} + \frac{\alpha}{2\Delta}\right)\right] \tag{4.1}$$

$$w^2 + \alpha w\bar r + \bar r^2 = W^2 \exp\left[\frac{\alpha}{\Delta}\operatorname{arctg}\left(\frac{w}{\Delta\bar r} + \frac{\alpha}{2\Delta}\right)\right]; \quad \Delta^2 = 1 - \frac{\alpha^2}{4}$$

The logarithmic spirals

$$\omega = V\exp\frac{\alpha\varphi}{2\Delta}, \quad \omega = W\exp\frac{\alpha\varphi}{2\Delta}$$

are obtained in the polar coordinates

$$\Delta\bar r = \omega\cos\varphi, \quad v + {}^1/_2\alpha\bar r = \omega\sin\varphi, \quad w + {}^1/_2\alpha\bar r = \omega\sin\varphi$$

The solution of system (2.4), adjoining a vacuum when $r = r_0$, $\bar q = \alpha r_0$ and $\bar c = 0$, corresponds to a curve in the (Q, C) plane which passes into the origin of the coordinate system. At the same time $v = w = 0$ and it follows from (4.1) that

$$V = W = r_0 e^{\beta}, \quad \beta = -\frac{\alpha}{2\Delta}\operatorname{arctg}\frac{\alpha}{2\Delta}$$

When $\bar r = 0$, this solution has the form

$$\bar q = r_0 e^{\beta}\operatorname{sh}\frac{\alpha\pi}{4\Delta} = q_0, \quad \bar c = c_0 = r_0 e^{\beta}\operatorname{ch}\frac{\alpha\pi}{4\Delta} \tag{4.2}$$

which corresponds to a point source.

Hence, a continuous flow from a point source into a vacuum corresponds to a curve joining the singularities. The solutions of (2.4) which depend on two parameters correspond to other integral curves of integral (2.6). They can be constructed with the help of the different branches of the functions v and w from (4.1) and describe flows which originate from point sources and are terminated by sinks.

Theorem 4. When $n = 1$, there exists a flow from a point source with arbitrary parameters into a vacuum which may pass through a shock wave.

Proof. If the source parameters q_0 and c_0 are identical to (4.2), the flow is continuous. In the general case of arbitrary parameters q_1 and c_1, solutions $L(c_1, q_1)$ of Eq.(4.1) are found such that $\bar q = q_1$, $\bar c = c_1$ when $\bar r = 0$. These curves are not terminated by a vacuum state.

Let us now consider the solutions $L(r_0, c_0, q_0)$ which are adjacent to a vacuum when $\bar r = r_0$. By virtue of (4.2), they depend on the single parameter r_0. Let us couple $L(r_0, c_0, q_0)$ with $L(c_1, q_1)$ when $\bar r = r_*$ with the aid of the condition for a shock transition (1.7). The equalities (1.7) become a system of two equations for determining the two unknowns r_0 and r_*.

When $n \neq 1$, there are four singular points of Eq.(2.6) in the half plane $C > 0$: two saddle points at the origin of coordinates and at an infinitely remote point and two foci at the points $C_\pm = \pm Q_\pm$, $Q_\pm = -{}^1/_2\alpha \pm ({}^1/_4\alpha^2 + 1/(n-1))^{1/2}$. Two integral curves emerge from the saddle points and pass into their own foci: one curve (L_+) emerges from $O = (0, 0)$ and passes to $F_+ = (Q_+, Q_+)$ while the other curve (L_-) emerges from $P(-\alpha, \infty)$ and passes to $F_- = (Q_-, -Q_-)$ (Fig.1). The remaining integral curves join the foci while filling the whole of the half plane.

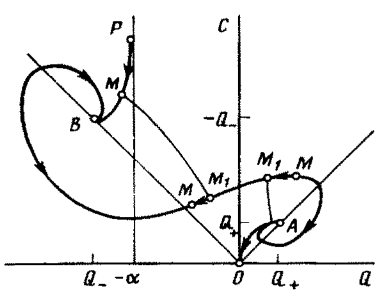

Fig.1

By virtue of (2.5), the variable $s = \ln\bar r$ increases along L_- upon moving from point P up to point B of the intersection of L_- with the straight line $C = -Q$ while it decreases along L_+ upon moving from point O up to point A of the intersection of L_+ with the straight line $C = Q$. In general, s changes the direction of increase on moving along any integral curve, if it intersects the curve $C = |Q|$. This means that the solutions of system (2.2) are not continuously extended beyond the curve $C = |Q|$.

The curve L_+, which passes into the saddle point O, has the asymptotic behaviour $Q = \alpha n^2(n+2)^{-1}C^2 + O(C^4)$ as $C \to 0$. The solution of system (2.4) is constructed using the principal part of this asymptotic form

$$\bar q = \alpha\bar r\left(\frac{3n+2}{n+2} - \frac{2n}{n+2}\frac{\bar r}{r_0}\right), \quad \bar r < r_0$$

$$\bar c \to 0, \quad d\bar c/d\bar r \to -\infty, \quad \bar r \to r_0$$

Consequently, the vacuum state corresponds to the singular point O.

The curve L_-, which emerges from the saddle point P, has the asymptotic behaviour $Q = -\alpha(1 + (n+2)^{-1}C^{-2}) + O(C^{-4})$ as $C \to \infty$. It follows from (2.4) that $\bar{q} \to 0 - 0$, $\bar{c} \to c_0$, $d\bar{c}/d\bar{r} \to 0 - 0$ as $\bar{r} \to 0 + 0$. Consequently, retardation corresponds to the singular point P.

The curves passing into the foci, $F_\pm$ have the asymptotic behaviour

$$C = Q_\pm + \frac{n-1}{2n}\left[\frac{1}{2}Q_\pm d\cos\theta \pm \left(\alpha Q_\pm \frac{n-3}{2} + \frac{2}{n-1}\right)\times \sin\theta\right] e^{\mp\delta(\theta-\theta_0)}$$

$$Q = \pm Q_\pm + e^{\mp\delta(\theta-\theta_0)}\sin\theta;\ \delta = \alpha(n-1)/d,$$
$$d = [32 - \alpha^2(n-3)^2]^{1/2}.$$

as $\theta \to \pm\infty$.

It follows from (2.4) that $\ln(r_{\pm\infty}/\bar{r}) = N_\pm e^{\mp\delta\theta}\sin(\theta - \theta_\pm)$ with certain constants $r_{\pm\infty}$, $N_\pm$, $\theta_\pm$. This means that $\bar{r} \to r_{\pm\infty} > 0$, $\bar{q} \to r_{\pm\infty}(Q_\pm + \alpha)$, $c \to \pm r_{\pm\infty}Q_\pm$ as $\theta \to \pm\infty$.

The solution $\bar{q}(\bar{r})$, $\bar{c}(\bar{r})$ of system (2.4) close to the foci, $F_\pm$, is multiple valued. The single-valued branches are constructed using the parts of the curves $L_\pm$ up to their intersections with the curve $C = |Q|$, that is, in fact, using the curves OA and PB. The boundaries $r_\pm$ of the domain, where the single-valued branches of the functions $\bar{q}(\bar{r})$ and $\bar{c}(\bar{r})$ are specified, are determined by integrating the first equation of (2.5) along the curves OA and PB. Hence, a continuous flow from a point source at $\bar{r} = r_+$ into a vacuum $\bar{r} = r_0$ corresponds to the curve OA while a continuous flow within a sphere $(n = 3)$ or a cylinder $(n = 2)$ with focussing at the centre of the sphere or on the axis of the cylinder corresponds to the curve PB.

Theorem 5. When $n > 1$, a flow exists from a point source with arbitrary parameters into a vacuum and this is perhaps through a shock wave and there exists a flow within a sphere (cylinder) with arbitrary input parameters with focussing at the centre (on the axis) and this may be through a shock wave.

Proof. If the input parameters are such that, in the variables Q and C, the data point lies on the curve $OA\,(PB)$, a continuous flow results. Suppose this is not satisfied. The velocity of the shock wave is equal to $\pm\bar{r}$. The equations of the shock transitions (1.7) in the variables Q and C have the form

$$(n+1)U = W_0,\quad (n+1)U = W;\quad U = Q + \alpha - (\pm 1) \tag{4.3}$$
$$U_0 = Q_0 + \alpha - (\pm 1),\quad W = nC^2U^{-1} + U,\quad W_0 = nC_0^2U_0^{-1} + U_0$$

If the source parameters $M = (P, Q)$ lie on an integral curve L of Eq.(2.6), which does not emerge from the saddle points, then the curve $W = F(U)$ is defined using it. Then, (4.3) defines the shock transition curve $(n+1)U_0 = F(W_0/(n+1))$. The points of intersection $M_0 = (P_0, Q_0)$, $M_1 = (P_1, Q_1)$ of the shock transition curve with the curves $OA\,(PB)$ and L define a flow with a shock wave. In order to find the place where the shock wave is located and the vacuum point (the focussing parameter), it is necessary to integrate one of the Eqs. (2.5) along a curve L from M to M_1 and along the curve L_+ from O to M_0 (L_- from P to M_0).

Remarks. 1°. The flows, which are investigated in paragraphs 3 and 4, occur in a bounded volume at a fixed t, unlike the conventional stationary solutions /3/ which spread out to infinity where a vacuum is attained. A constant stationary flow is only possible in the one-dimensional case $(n = 1)$.

2°. The solution of system (2.2) has a physical meaning for any natural n. A centro-symmetric solution in an n-dimensional space is considered in a three-dimensional subspace, $\bar{r}_i = 0$, $i \neq 1, 2, 3$. By virtue of the symmetry, $\bar{u}_i = 0$ and it follows from (1.2) that $u_i = 0$ and a solution of system (1.1) is obtained in the three-dimensional case with $\gamma = (n+2)/n$, where n is any natural number.

5. Functionally-invariant solutions. The solutions (2.1) of system (1.3) when $\alpha = \beta = \gamma = 0$ and $U = W = 0$ are functionally invariant, that is, they depend on arbitrary functions. Actually, system (1.3) takes the form

$$P_{\bar{s}} = R(V^2\bar{s}^{-1} - \bar{s}),\quad P_{\bar{z}} = -\bar{z}R \tag{5.1}$$

If the entropy is constant, that is, $S = S_0$, $P = A(S_0)R^{(n+2)/n}$, $A > 0$, then $V_{\bar{z}} = 0$, $\bar{R} = -\mu + \varphi(\lambda)$, $V^2 = \lambda(\varphi' + 1)$, where $\bar{R} = (n+2)AR^{(n+2)/n}$, $\mu = \bar{z}^2$, $\lambda = \bar{s}^2$ and $\varphi(\lambda)$ is an arbitrary function but such that $\varphi \geqslant 0$, $\varphi' \geqslant -1$ in the flow domain. The boundary with a vacuum is specified by the equation $\mu = \varphi(\lambda)$.

If the entropy is not constant, then $R = -2P_\mu$, $V^2 = \lambda(1 - P_\mu^{-1}P_\lambda)$ where $P(\lambda, \mu)$ is an arbitrary function but such that $P_\lambda \leqslant P_\mu < 0$ in the flow domain. The equation $P_\mu = 0$ defines the boundary with a vacuum.

The surface of a contact discontinuity is defined from the equality $[P] = 0$. The remaining conditions (1.6) are satisfied by virtue of the equality $u_n = 0$. Hence, any two solutions of (5.1) with intersecting flow domains are coupled along a surface of the same pressure through a contact discontinuity.

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EFFECT OF SPHERICALLY SYMMETRIC MASS FLOW FROM THE SURFACE OF A PARTICLE ON THE FORCE OF INTERACTION WITH A PLANE SURFACE*

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A stationary velocity field of the flow of a gaseous medium generated by uniform radial injection from the surface of a spherical particle near a wall is considered in the Stokes' approximation. Bispherical coordinates are used to write the expression for the stream function. A formula is obtained for the force acting on the spherical particle when there is an arbitrary mass flow from its surface, generalizing earlier results /1, 2/. An expression for the force acting on the particle is obtained for the case of spherically symmetric injection from the surface of the particle, and asymptotic formulas at short and long distances from the wall are studied.

An analogous problem concerning the forces of interaction between two spherical particles of the same radius, when uniform injection of equal intensity takes place from their surfaces, is discussed. This is equivalent to the problem of the interaction of a spherical particle with a free surface. A general expression for the force of interaction, and its asymptotic forms for short and long distances, are obtained.

1. Formulation of the problem. Evaporation from a spherical particle near a solid or free surface, caused by various processes taking place in the gaseous medium, at the surface and inside of the particle, can be regarded in certain cases as being close to spherically symmetric.

Let us consider, for example, a particle with internal heat emission, situated near a wall at a uniform temperature T_w, equal to the temperature of the gaseous medium far from the particle. We shall assume that the heat flux from the surface of the particle is governed

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